

A half-automated study of a 2-parameter family of integrals

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Abstract

The study of some parametric integrals is presented with a combined approach of analytical development, the usage of a Computed Algebra System (CAS) and of the Online Encyclopedia of Integer Sequences. The paper runs around the analysis of the 2-parameter definite integral

$$I_n^{(p)} = \int_0^{\pi/4} x^p \tan^n x \, dx.$$

The methodology for the solution includes a) an analytical investigation for the study of the parametric integral, b) computations with a CAS of the integral for specific values of the parameter, c) investigation of the connection between the integral and special functions or mathematical constants, and d) derivation of a general algorithm for the complete computation of the parametric integral. Comparing the outputs of different kinds of software yields useful remarks about the understanding of the outputs, an important issue in mathematics education in a technology-rich environment.

1 Introduction.

The computation of parametric definite integrals is an interesting mathematical field (e.g. see [27, 7]). On the one hand, parametric integrals help students to grasp more abstract situations than in the cases without parameters. On the other hand, in domains such as Physics and Engineering, numerous phenomena have a mathematical translation into such integrals; for example, see [17]. Moreover, such computations may lead to formula for indefinite series, and also to derive combinatorial identities and integral representations for combinatorial objects; for example, see [12, 15, 6, 16, 28, 29, 32] and the papers in reference there. In the specific case of Catalan numbers, different presentations and applications are given in [24].

Computing the integrals by hand may be unilluminating. The usage of technology has been analyzed and discussed for a long time, not only for the domain that we are interested in here. Buchberger [9] says that a Computer Algebra System (CAS) should be used only where the user knows how to perform the computations by hand, not as a black box where the user has no knowledge on what are

the processes at work inside. A good example of this approach is in the book [2] where the described algorithms are first "run by hand", and then applied with their version implemented in software¹. With a black box usage, the CAS is a facilitator to obtain a result, not to develop more mathematical knowledge. For a short presentation of this *White-Box / Black-Box Principle*, we refer to [10].

Nevertheless, but as already seen in [13], it happens that the CAS is used to bypass a lack of theoretical knowledge, then proceeding backwards the gap can be filled. In [19], Drijvers notes that generally a CAS works as a black box, it does not provide insight into the algorithms. Advanced versions of a CAS may provide some insight, either as a step-by-step command, as in the Derive software (in version 6.1, released a long time ago, which was the last release), or via tutorials such as Maple's tutorials. These tutorials work in an interactive way, providing hints for every single step, sometimes more than one. The user then chooses and checks the efficiency of the proposed step. A suitable usage of such a tutorial provides a good scaffolding to the user, which has to be gradually faded. These tutorials are a kind of intelligent tutoring systems, an interesting AI application which can propose hints, analyze the user's answer, monitor and guide students (Kock et al. [23] compare this with the generative AI systems which became popular since year 2023 with the first launch of ChatGPT). Note that the same tutorials have a shortcut giving the final answer immediately; this is a black box usage. Anyway, this usage is highly personal, and each user will develop his/her own instrumental genesis [3, 31]. A good balance between paper-and-pencil and CAS assisted work is crucial. As we will see, "the computer offers to provide scaffolding both to enhance mathematical reasoning and to restrain mathematical error" (J. Borwein in [8]).

In this paper, we study a 2-parameter family of definite integrals. For small values of the parameters, computations may be performed by hand, but this is time-consuming. A CAS can be used after a few computations, without contradicting Buchberger's point of view. This may be a partial solution only, as we will see in Section 2. In certain cases, the output of an automated computation shows an integral which cannot be computed analytically. From that point, either numerical methods should be used, or an induction formula has to be looked for, as in [15, 16] and other cases referenced there.

The numerical results obtained with a CAS have to be exploited. The investigation made a strong usage of the Online Encyclopedia of Integer Sequences (OEIS)². This proved a really efficient approach, enabling to enounce conjectures, which have then to be proven.

The integral

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

and an application to soil mechanics have been described in [17]. Here we study a generalization of it, namely the 2-parameter integral defined by:

$$I_n^{(p)} = \int_0^{\pi/4} x^p \tan^n x \, dx, \quad (1)$$

where the parameters p and n are non negative integers.

A motivation has been provided by students of the 2nd author in an engineering curriculum, who came to him from time to time to ask for additional exercises, beyond the official syllabus. It is

¹The book is devoted to Gröbner bases and their applications. This is not the topic of our work here, but the didactic methodology is similar to what we claim

²<http://www.oeis.org>.

valuable to be prepared to answer requests by highly motivated students; the authors had opportunities to propose such extensions of the syllabus. They are involved both in mathematics teachers education and in engineering education. Even when the syllabi do not mention explicitly teaching and working in a technology-rich environment, they think that this is what has to be done nowadays, inciting students to develop their own technological skills, preparing them also for lifelong learning. Such exercises have been included in practice sessions of a Calculus course at JCT.

First, we derive by hand an induction formula, then we try to have more insight into the 2-parameter sequence of integrals. For this we use a Computer Algebra System, here the Maple package and its tutorials, and the Online Encyclopedia of Integer Sequences.

Automated methods have been analyzed and documented for questions in Geometry, Analytic Geometry and in Differential Geometry (for example, see the survey [14]). The important novelty here is that, for parametric integrals no visual intuition is afforded. This is not rare with integrals, recall the example of $\int x \ln x \, dx$ for which the choice of the functions for an integration by parts is counter-intuitive. The present study requests more abstract thinking. After all, parametric integrals are already a more abstract object than "ordinary" definite integrals.

2 An induction formula

For integrals where the integrand contains a power of the tangent function, it is natural to look for an induction with an increment of 2 in the power. We have:

$$\begin{aligned} I_n^{(p)} + I_{n-2}^{(p)} &= \int_0^{\pi/4} x^p \tan^n x \, dx + \int_0^{\pi/4} x^p \tan^{n-2} x \, dx \\ &= \int_0^{\pi/4} x^p (\tan^n x + \tan^{n-2} x) \, dx \\ &= \int_0^{\pi/4} x^p (\tan^2 x + 1) \tan^{n-2} x \, dx \end{aligned}$$

Integration by parts of the last expression, using the functions:

$$f(x) = x^p \text{ and } g(x) = \frac{\tan^{n-1} x}{n-1} \quad (2)$$

whence

$$f'(x) = px^{p-1} \text{ and } g'(x) = \tan^{n-2} x (\tan^2 x + 1) \quad (3)$$

This leads to the following recurrence relation:

$$I_n^{(p)} + I_{n-2}^{(p)} = \frac{1}{n-1} \frac{\pi^p}{4} + \frac{p}{n-1} I_{n-1}^{(p-1)}, \quad p \geq 1; \quad n \geq 1. \quad (4)$$

We can also derive general recurrence formulas for both the sum $\sum_n I_n^{(p)}$ for a given p , and for $I_n^{(p)}$.

Proposition 1 When n is even, we denote $n = 2l$, and the following holds:

$$I_{2l}^{(1)} = \frac{\pi}{4} \sum_{k=1}^{2l-1} \frac{(-1)^k}{k} + \sum_{k=1}^{2l-1} \frac{(-1)^k I_k}{k} - \frac{1}{2} \frac{\pi^2}{4} \quad (5)$$

2.1 The case of even n

2.1.1 Analytic work

For a given integer $p \geq 1$ and if n is even:

$$\begin{aligned} I_n^{(p)} + 2(I_{n-2}^{(p)} + I_{n-4}^{(p)} + \cdots + I_2^{(p)}) + I_0^{(p)} &= \frac{\pi}{4} \sum_{k=1}^{n-1} \frac{1}{k} + p \sum_{k=1}^{n-1} \frac{I_k^{(p-1)}}{k} \\ &= (\Psi(n) + \gamma) \frac{\pi}{4} + p \sum_{k=1}^{n-1} \frac{I_k^{(p-1)}}{k}. \end{aligned}$$

By addition and subtraction of the iterative formula in Equation (4), we derive the following formula:

$$I_n^{(p)} + I_0^{(p)} = \frac{\pi}{4} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + p \sum_{k=1}^{n-1} \frac{(-1)^k I_k^{(p-1)}}{k}. \quad (6)$$

The first integral in the sequence is

$$I_0^{(p)} = \frac{1}{p+1} - \frac{\pi}{4} \quad (p+1), \quad (7)$$

therefore, the following holds:

$$I_n^{(p)} = \frac{\pi}{4} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + p \sum_{k=1}^{n-1} \frac{(-1)^k I_k^{(p-1)}}{k} - \frac{1}{p+1} - \frac{\pi}{4} \quad (p+1) \quad (8)$$

Equation (8) means that $I_n^{(p)}$ is a linear function of the previous integrals $I_n^{(k)}$, for $k < p$.

If n is even and $p = 0$, we obtain the following identity:

$$I_n^{(0)} + I_0^{(0)} = \sum_{k=1}^{n-1} \frac{I_k}{k}. \quad (9)$$

Using the previous derivation, we obtain $I_0^{(0)} = \frac{\pi}{4}$, whence:

$$I_n^{(0)} = \sum_{k=1}^{n-1} \frac{I_k}{k} - \frac{\pi}{4}. \quad (10)$$

If n is even and $p = 1$, we obtain:

$$I_n^{(1)} + 2(I_{n-2}^{(1)} + I_{n-4}^{(1)} + \cdots + I_2^{(1)}) + I_0^{(1)} \quad (11)$$

Also a general expression of $I_n^{(1)}$ may be derived:

$$I_n^{(1)} = \frac{\pi}{4} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + \sum_{k=1}^{n-1} \frac{(-1)^k I_k}{k} - \frac{1}{2} - \frac{\pi}{4} \quad (12)$$

2.1.2 Analysis with a CAS

Using a Computer Algebra System, we obtain the following values for small even values of the parameter n :

$$\begin{aligned} I_0^{(1)} &= \frac{\pi^2}{32} \\ I_2^{(1)} &= -\frac{\pi^2}{32} + \frac{\pi}{4} - \frac{\ln 2}{2} \\ I_4^{(1)} &= \frac{\pi^2}{32} - \frac{\pi}{6} - \frac{1}{6} + \frac{2}{3} \ln 2 \\ I_6^{(1)} &= -\frac{\pi^2}{32} + \frac{13\pi}{60} + \frac{13}{60} - \frac{23}{30} \ln 2 \\ I_8^{(1)} &= \frac{\pi^2}{32} - \frac{19\pi}{105} - \frac{29}{105} + \frac{88}{105} \ln 2 \\ I_{10}^{(1)} &= -\frac{\pi^2}{32} + \frac{263\pi}{1260} + \frac{2333}{7560} - \frac{563}{630} \ln 2 \\ I_{12}^{(1)} &= \frac{\pi^2}{32} - \frac{1289\pi}{6930} - \frac{3578}{10395} + \frac{3254}{3465} \ln 2 \\ I_{14}^{(1)} &= -\frac{\pi^2}{32} + \frac{36979\pi}{180180} + \frac{397753}{1081080} - \frac{88069}{90090} \ln 2 \end{aligned}$$

Remark 2 Look at the coefficient of π in this output. It determines 2 sequences of integers, a sequence of numerators 1, -1, 13, -19, 263, ... and a sequence of denominators 4, 6, 60, 105, 1260, The database OEIS does not have an entry for the sequence of denominators, but proposes to consider two subsequences according to the index being either odd or even. Starting from this database searching, some non trivial connections can be found between the integrals studied here and other mathematical objects, with an approach similar to [12, 15]. Such a study is not in the scope of the present paper, we leave it as an open question for a later study.

2.2 The case of odd n

2.2.1 Analytic work

For a given integer $p \geq 1$ and odd n , we have:

$$I_n^{(p)} + 2 I_{n-2}^{(p)} + I_{n-4}^{(p)} + \cdots + I_3^{(p)} + I_1^{(p)} = \frac{\pi}{4} \sum_{k=2}^{n-1} \frac{1}{k} + p \sum_{k=2}^{n-1} \frac{I_k}{k}$$

By addition and subtraction of the iterative formula (i.e., Equation (4)), we obtain:

$$-I_n^{(p)} + I_1^{(p)} = \frac{\pi}{4} \sum_{k=2}^{n-1} \frac{(-1)^k}{k} + p \sum_{k=2}^{n-1} \frac{(-1)^k I_k^{(p-1)}}{k}$$

For $p = 0$, we find the previous expression. For even n and $p = 1$, we obtain:

$$I_n^{(1)} + 2(I_{n-2}^{(1)} + I_{n-4}^{(1)} + \cdots + I_3^{(1)}) + I_1^{(1)} = \frac{\pi}{4} \sum_{k=2}^{n-1} \frac{1}{k} + \sum_{k=2}^{n-1} \frac{I_k}{k}$$

Then we derive a general expression for $I_n^{(1)}$ and $n = 2l + 1$:

Proposition 3

$$I_{2l+1}^{(1)} = \frac{\pi}{4} \sum_{k=2}^{2l} \frac{(-1)^k}{k} + \sum_{k=2}^{2l} \frac{(-1)^k I_k}{k} - I_1$$

2.2.2 Analysis with a CAS

Using a Maple, we obtain the following values for small odd values of the parameter n :

$$\begin{aligned} I_1^{(1)} &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx \\ I_3^{(1)} &= -\frac{\pi}{8} \ln 2 - \int_0^{\pi/4} \ln(\cos x) dx + \frac{\pi}{4} - \frac{1}{2} \\ I_5^{(1)} &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx - \frac{\pi}{4} + \frac{2}{3} \\ I_7^{(1)} &= -\frac{\pi}{8} \ln 2 - \int_0^{\pi/4} \ln(\cos x) dx + \frac{\pi}{3} - \frac{73}{90} \\ I_9^{(1)} &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx - \frac{\pi}{3} + \frac{284}{315} \\ I_{11}^{(1)} &= -\frac{\pi}{8} \ln 2 - \int_0^{\pi/4} \ln(\cos x) dx + \frac{23\pi}{60} - \frac{3103}{3150} \\ I_{13}^{(1)} &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx - \frac{23\pi}{60} + \frac{54422}{51975} \\ I_{15}^{(1)} &= -\frac{\pi}{8} \ln 2 - \int_0^{\pi/4} \ln(\cos x) dx + \frac{44\pi}{105} - \frac{10459489}{9459450} \\ I_{16}^{(1)} &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx - \frac{44\pi}{105} + \frac{5452712}{4729725} \end{aligned}$$

The following remarks can be made:

1. Two subsequences appear, for even indices and for odd indices.
2. The integral $\frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\cos x) dx$ appears for an index $n \equiv 1 \pmod{4}$, and its opposite for $n \equiv 3 \pmod{4}$. The sign change comes from the $(-1)^k$ factor in the two first terms in Proposition (3).
3. The integral $\int_0^{\pi/4} \ln(\cos x) dx$ is left in closed form and cannot be evaluated analytically.

With Maple, the following result is obtained:

$$\int_0^{\pi/4} \ln(\cos x) dx = \frac{\pi}{4} \ln 2 + \frac{1}{2} G, \quad (13)$$

where G denotes the Catalan constant³.

In [1], the integral presentations of the Catalan constant are given, but without analytic proofs. There, computations have been performed with Mathematica. Note that this is a black-box usage of the CAS, in contradiction to Buchberger's point of view. Nevertheless, we can follow a path in reversed direction and note that

$$G = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2}. \quad (14)$$

3 Study of the influence of the parameter p

In this section we investigate the dependence of the parameter p on the 2-parameter integral $I_n^{(p)}$. The fundamental iterative formula (4) connects $I_n^{(p)}$ with the terms $I_{n-2}^{(p)}$ and $I_{n-1}^{(p-1)}$ and this for a given p .

The influence of p may be checked in two ways.

First, in the given integration interval, i.e. for $0 \leq x \leq \frac{\pi}{4} < 1$, we have $0 < x^p < 1$. It follows that $I_n^{(p)} < I_n$, where

$$I_n = \int_0^{\pi/4} \tan^n x \, dx. \quad (15)$$

3.1 The tangent power integral

This last equation has been studied by the authors in [17]. In this reference, we derived closed combinatorial formulas for the tangent-power integral, according to the congruence class modulo 4 of the parameter m . We presented three different forms of the definite integral:

1. A finite series formula: for even indices, the following holds:

$$I_{2k} = (-1)^k \frac{\pi}{4} + \sum_{l=1}^k \frac{(-1)^{(l+k)}}{2l-1}. \quad (16)$$

For odd n , we have:

$$I_{2k+1} = (-1)^k \frac{\ln 2}{2} + \sum_{l=1}^k \frac{(-1)^{(l+k)}}{2l}$$

2. An equivalent formula using double factorials:

$$\begin{aligned} I_{4k-2} &= -\frac{\pi}{4} + \frac{k}{(2k-1)!!} \\ I_{4k-1} &= \frac{\ln 2}{2} + \frac{1}{2} + \frac{k}{(2k-1)!!} \\ I_{4k} &= \frac{\pi}{4} + \frac{k}{(2k-1)!!} \\ I_{4k+1} &= \frac{\ln 2}{2} + \frac{1}{2} + \frac{k}{(2k)!!} \end{aligned}$$

³See the sequence <https://oeis.org/A006752>

These different expressions have different physical meanings leading to different types of understandings and teaching:

3.2 An iterative algorithm for $I_n^{(p)}$ for n and $p = 1, \dots, k$

In the previous section we derived a general iterative equation for fixed n between $I_n^{(p)}$ and the terms $I_k^{(p-1)}$, $k < n$:

- If n is even:

$$I_n^{(p)} = \frac{\pi}{4} {}^p \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + p \sum_{k=1}^{n-1} \frac{(-1)^k I_k^{(p-1)}}{k} - \frac{1}{p+1} \frac{\pi}{4} {}^{(p+1)}$$

- If n is odd:

$$-I_n^{(p)} + I_1^{(p)} = \frac{\pi}{4} {}^p \sum_{k=2}^{n-1} \frac{(-1)^k}{k} + p \sum_{k=2}^{n-1} \frac{(-1)^k I_k^{(p-1)}}{k}$$

As a consequence, we may compute iteratively any integral $I_n^{(p)}$. The method consists in:

- (i) Check if n is even or odd
- (ii) If n is even, compute $I_k^{(0)} = \int_0^{\pi/4} \tan^n x \, dx$ for $k < n$.
- (iii) Then use the above formula to compute $I_n^{(1)}$
- (iv) Then compute $I_k^{(1)}$ for $k < n$.
- (v) Compute $I_n^{(2)}$ and so on until $I_n^{(p)}$
- (vi) If n is odd, compute $I_k^{(0)} = \int_0^{\pi/4} \tan^n x \, dx$ for $1 < k < n$.
- (vii) Then use the above formula to compute $I_n^{(1)}$
- (viii) Then compute $I_k^{(1)}$ for $k < n$.
- (ix) Compute $I_n^{(2)}$ and so onuntil $I_n^{(p)}$

3.3 Analysis of an integral using a power series

We consider the integral

$$J_n = \int_0^{\pi/4} \frac{\tan^n x}{1-x} dx. \quad (17)$$

For any integer $n > 0$, we were unable to obtain a direct analytic computation of this integral using a CAS. Only numerical values have been obtained. Nevertheless, the integral J_n may be computed

using the integral $I_n^{(p)}$ studied earlier. The domain of integration is the interval $[0, \frac{\pi}{4}]$, therefore $|x| < 1$ and we have

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i,$$

and it follows that

$$J_n = \int_0^{\pi/4} \frac{\tan^n x}{1-x} dx = \sum_{i=0}^{\infty} \int_0^{\pi/4} x^i \tan^n x dx = \sum_{i=0}^{\infty} I_n^i.$$

The computation of J_n requires the computation of the integrals I_n^i for all i . In the previous section, we presented an iterative method to compute I_n^i from the different I_n^{i-1} ; $i \geq 1$. Now we present an approximate method to compute J_n .

We have:

$$\forall n \geq n_0, I_n^{(i)} \approx \left(\frac{\pi}{4}\right)^i \int_0^{\pi/4} \tan^n(x) dx.$$

It follows that:

$$J_n = \sum_{i=0}^{\infty} I_n^{(i)} = \sum_{i=0}^{n_0} I_n^i + \sum_{i=n_0}^{\infty} I_n^{(i)}.$$

Using the above approximation, we obtain:

$$J_n = \sum_{i=0}^{n_0} I_n^i + \left[\sum_{i=n_0}^{\infty} \left(\frac{\pi}{4}\right)^i \right] \left[\int_0^{\pi/4} \tan^n(x) dx \right].$$

Therefore, we obtain:

$$J_n = \sum_{i=0}^{n_0} I_n^i + \left[\left(\frac{\pi}{4}\right)^{n_0} \frac{1}{1 - \frac{\pi}{4}} \right] \int_0^{\pi/4} \tan^n(x) dx.$$

The index n_0 may be chosen by using an error level ϵ and require that $x^{n_0} < \epsilon$.

3.4 Analysis of the integral $L_n = \int_0^1 \arctan^n(x) dx$

Note first that this integral is a particular case of the integral $I_n^{(p)}$ studied above. When asked to compute this integral for small values of the parameter, a CAS returns only the closed form of the integral. Only in numerical form, some other output can be obtained, but this is irrelevant to our purpose here..

Consider the change of variable: $u = \arctan(x)$; then $x = \tan(u)$ for $0 \leq u \leq \frac{\pi}{4}$. Note that $dx = (1 + \tan^2 u) du$. With this change of variable $L_n = \int_0^{\pi/4} u^n (1 + \tan^2 u) du$, whence:

$$L_n = \int_0^{\pi/4} u^n du + I_2^{(p)} \quad (18)$$

Finally, we have the following proposition:

Proposition 4 For every non negative integers n and p , the following holds:

$$L_n = \frac{1}{n+1} \left(\frac{\pi}{4}\right)^{n+1} + I_2^{(p)}.$$

4 Some thoughts in the aftermath

Previous explorations of parametric definite integrals, no matter how many parameters were involved, opened for us various ways to build bridges between mathematical domains. In many cases, integral presentations of combinatorial objects have been derived. In other cases, they provided an opportunity to prove combinatorial identities [15, 16, 28, 29]. A CAS is an important tool for such an exploration and proofs. Nevertheless, it is sometimes hard to conjecture such an identity and another technology can be useful. This was the case for the mentioned works, and using the Online Encyclopedia of Integer Sequences revealed efficient. Of course, it helped to have a conjecture, but this conjecture had to be proven by more theoretical means.

For the examples of the present work, the situation is somehow different. We could try various inputs for searching the database, but none provided a ready-to-use answer. For all our trials, the database proposed a modified version of the input, and understanding the proposed sequences was non-trivial. This is a good illustration that exploring mathematical objects in a technology-rich environment does not follow a predefined path.

The OECD defined the so-called 4 C's of 21st century education: Communication, Collaboration, Critical Thinking and Creativity. These are crucial in various settings (see for example [30]). Here, the collaboration can be between humans, but also between man and machine. Collaboration with technology, i.e. human and machine together, requires strong Critical Thinking in order to analyse the outputs of the CAS and of the interactive database. We shall mention that collaboration with a generative AI requires this still more; see [4, 23]. Critical Thinking and Creativity are central skills, opposite to Black-Box usage of technology. Further, Creativity leads to the possible conjectures. This is a modern way to deal with mathematics, with a scheme exploration-conjecture-proof. It request also the understanding that the new technological skills are an integral part of the new mathematical knowledge, and that a new technological discourse has to be developed [3, 26]. Such exploration can be initiated by a 5th C, namely Curiosity; a kind of exploration has been proposed in Remark 2. This has been the basis of [17], where mathematical curiosity and communication and collaboration between researchers yielded an interesting application.

Nowadays, curiosity can push in new directions. We asked an AI for references about parametric definite integrals. Most answers gave general chapters in Calculus books, the only relevant to *parametric* definite integrals was a presentation in a conference [18].

Finally, we wish to recall what we wrote in Section 1. Parametric definite integrals enable students to step forwards towards more abstract understanding than what is offered in a first Calculus course. But they are not always a pure mathematical exercise. They have numerous applications in science, in particular in Physics; [22] provides an insight into students' difficulties to translate the knowledge acquired in definite integrals in a Calculus course into the needed skills for Physics. Nevertheless, the topics is quite rare in Calculus textbooks, and also in the classical syllabus; we found a subchapter in [25] (subchapter 7.4). Despite the fact that these applications are not an integral part of the official syllabus, the authors try to give by that way some motivation to the students asking "what is this good for?". Regarding the mental schemes that they require, these are parallel to what is required in Linear Algebra for solving parametric systems of linear equations; see [11], where some problems are analyzed, such as the lack of investigation of special values of the parameters⁴.

⁴Such problems, when working with AI for mathematics education, are analyzed in [4].

Appendix: the work with mathematical software The exploration of the sequence of parametric integrals on subsubsections 2.1.2 and 2.2.2 has been performed using two different softwares. As mentioned above, we used Maple 2024, but also Derive⁵. This last is not available anymore, but the community of users still exists. Even if all of the members use now other packages, they still share applications during conferences, and/or use it in class.

The formulas displayed in subsubsections 2.1.2 and 2.2.2 have been obtained with Derive, as shown in Figure 1. The 2-parameter family of integrals is defined, then it is computed for small values of the parameters. For this, the **Vector** command is applied. Its syntax is **Vector(parametric formula, parameter, starting value, end value, step)**. The choice of the starting value and the step enables to distinguish what happens with odd (resp. even) values of the parameter. A noticeable feature of the output is the presence of a closed integral, namely $\int_0^{\pi/4} \ln(\cos x) dx$.

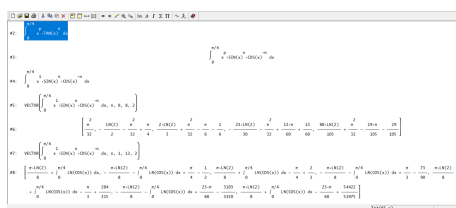


Figure 1: A session with Derive

We are aware that a software discontinued 20 years ago cannot be recommended. Maybe the example here (as in previous published works [?]) will incite development of such an output and make it available in existing software.

In parallel, we used the following Maple code (of course, with different choices for the values of the parameter and step):

```
restart;
f := x*tan(x)^n;
for i from 0 by 2 to 14 do
    n := i;
    int(f, x = 0 .. Pi/4);
end do;
```

In order to take into account the 2 parameters, we used nested **for** loops. The output appears different from above. For odd values of the parameter, it involves the so-called *Catalan constant*, as shown in Figure 2. Actually, the answers by the two packages are equivalent, as explained by Equation (13). It has been checked also with Maple, as shown in Figure 3.

This is a good opportunity to draw students' attention to the fact that they should not relate to the output as THE answer, but rather as a representation of the answer. It is well known that mathematical objects cannot be grasped with hands, but are dealt with using various representations. A rich knowledge is developed when switching between different representations, which can reveal different aspects of the same mathematical object; see [20]. Here we have, on the one hand, a representation

⁵We have this opportunity to evoke Josef Böhm's memory, a good friend devoted to education and to the Derive community; he was the editor of the Derive Newsletter.

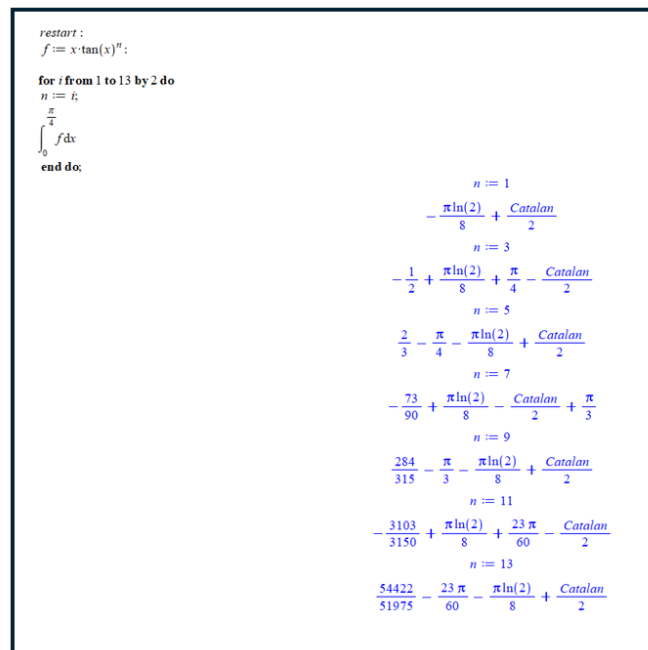


Figure 2: A session with Maple

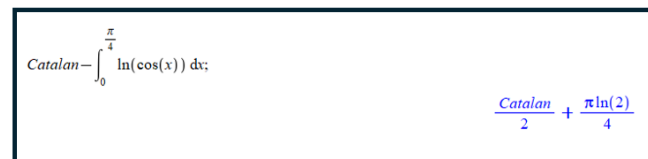


Figure 3: Verification that the two outputs are equivalent

using an implicit form for a definite integral, in a situation where the integrand has no primitive being described by a closed formula. Generally, the registers of representation mentioned for mathematical objects are graphical, numerical, symbolic, etc. The objects under study in this work belong to Calculus, and the representations that we dealt with may be explained as belonging to a symbolic registers, but their appearances are different. Of course, engineers may wish to have a numerical representation, which can be more easily applied in concrete situations. Maybe we should speak here about subregisters of representation. Such a situation is not rare, and students meet them quite early. On the other hand, we have a representation involving the Catalan constant, an object which is rarely met by undergraduates. This provides an opportunity for a small extension of the curriculum.

Finally we wish to mention that the reflections about the different roles of technology usage for helping humans address mathematical issues do not need to apply only to students, they are equally useful for researchers.

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